Buffy Cashell was a junior in Perrysville High School when this paper was written for the Ohio Academy of Science (It was rated SUPERIOR). Buffy welcomes your comments.

THE DISPROOF OF ZENO'S ARROW PARADOX

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PROBLEM

Zeno's Arrow Paradox states that, at any instant in time, an arrow in motion will actually be in a fixed position if the segment of time is made small enough. Thus an object being in a fixed position at every instant can never move. This paradox has puzzled great mathematicians and philosophers since the fifth century B.C. including, Aristotle, Thomas Aquinas, Descartes, Leibniz, Spinoza, Bergson, Weierstrass, Cantor, Bolzano, and Berkeley. They were unable to prove or disprove Zeno's paradox using knowledge available in their day (2).

HISTORY

When early philosophers and mathematicians began thinking about infinity, they encountered several problems. Two concepts of infinity emerged, the infinitely large and the infinitely small. The infinitely large was the easiest to comprehend. The simple expression n+1 enabled them to accept the concept of infinitely large because they could continue expanding the largest number that they could think of with the number n+1. The infinitely small, however, proved extremely hard to grasp (1).

The concept of infinitely small led to a difficult series of proofs, one of the most famous being a paradox by Zeno called The Dichotomy which states that it is impossible to cover any given distance (7). The argument is that half of the distance must be traversed, then half of the remaining distance, then half of what remains, and so forth. It states that some portion of the distance to be covered always remains; therefore, motion is impossible.

The following infinite geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} \ldots$

represents the distance covered.

Each term is half of the value of the one before it. Although the series is infinite, it has a finite sum, the number 1. This illustrates the incorrect assumption by Zeno in his Dichotomy (6). Zeno assumed that since a distance is composed of an infinite number of parts that the distance must be infinite (2).

The infinitesimal still presents a problem. It does not equal zero, but it is smaller than any quantity. At the same time, a sizeable amount of the infinitesimal can combine to make a very definite amount. To deal with the idea that the infinitesimal never reaches a number, but approaches it infinitely, philosophers and mathematicians developed the law of limits. The law of limits simply explains that a series can approach a number and never reach it.

PROCEDURE

The idea for this project originated from reading about Zeno's Arrow Paradox in the book, The Road To Infinity, by Issac Asimov. The first attempt was to prove the paradox to be true. The next avenue of investigation developed when another professor sent information that dealt with the paradox. But a professor advised that a disproof seemed to be a more logical approach. In order to disprove this paradox the researcher must prove that there is motion. Attacking the problem from that direction for several weeks produced no results. Eventually it was necessary to define motion, then move back to the initial problem. An appropriate definition of motion contains the expressions non-zero acceleration component and non-zero velocity component. The disproof needed to show that the arrow contained a non-zero acceleration component and a non-zero velocity component. Two ideas emerged, first that the force of gravity acting upon the arrow can be used to express the non-zero acceleration component. Force due to gravity equals mass x acceleration due to gravity (which is 32ft./sec.) (8). (We assume the mass of an arrow does not equal zero.) Secondly, the momentum of the arrow can be used to express the non-zero velocity component, i.e. momentum equals mass x velocity (8). The velocity does not equal zero until the arrow reaches the ground.

ZENO'S ARROW PARADOX: WAS ZENO WRONG?

Begin by assuming that Zeno is correct and that the arrow is stationary at some time. Using present-day knowledge of physics and mathematics concerning motion on earth, we may conclude that for the arrow to be actually stationary at any instant in time, it must have counterbalancing forces acting upon it in all directions at that instant in time (8). We know empirically, however, that the arrow is in another position an instant of time later and that this position is forward from its previous position.

It can be seen, then, that at least two forces (downward-gravity, forward-momentum) are acting upon it to enable it to move to the next position. The downward force we know from physics to be gravity (8). The force of gravity is always acting on the arrow in the downward direction. We also observe that the arrow has moved forward from position to position. There is no additional horizontal force added to the arrow at or immediately beyond its first position, therefore, there must be an existing force acting on the arrow in the forward (horizontal) direction at that point.

If indeed Zeno was correct and the arrow is stationary at some time, then we must also assume that the arrow has no unbalanced forces acting upon it. We have shown that at least two unbalanced forces must be acting on the arrow at every instant the arrow is in flight and that these unbalanced forces cause the arrow to move in a forward and downward direction. From this we may conclude that the arrow must be in motion at every instant that it is in flight, regardless of the division of time used to isolate the arrow in its path for study. Thus, using a combination of modern day knowledge of mathematics and physics, simple logic, and some empirical observations, it can be shown that Zeno's Arrow Paradox is not really a paradox after all.

Works Consulted

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A THEOREM ABOUT NINES

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This theorem talks about the square of a number of 9's, such as 9^2, 99^2, 999^2. If you punch a number for squaring in a calculator of more than five 9's, then it turns out as an expansion number [scientific notation]. Everybody knows $9^2 = 81$, $99^2 = 9801$, $999^2 = 998001$, but not many people know $9999999999^2 = 99999999980000000001$. It is easy to see the sequence. They just repeat 9,8,0,1. If n = the number of 9's, then the formula is (n-1) 9's, 8, (n-1)0's, 1.

To prove $9999^2 = 99980001$: $9999^2 = (10000\text{-}1)^2 = 10000^2 - 2\cdot 10000 + 1 = 100000000 - 20000 + 1 = 99980001$. Likewise: $(10^n - 1)^2 = 10^{2n} - 2\cdot 10^n + 1 = 1\underbrace{0\ldots0}_{2n\text{ zeros}} - 2\underbrace{0\ldots0}_{n\text{ zeros}} + 1 = \underbrace{9\ldots9}_{(n\text{-}1)9\text{'s}}\,8\,\underbrace{0\ldots0}_{n\text{ zeros}} + 1 = \underbrace{9\ldots9}_{(n\text{-}1)9\text{'s}}\,8\,\underbrace{0\ldots0}_{(n-1)\text{ zeros}}\,1.$

Teacher's note: Jung-mee Kim had been a student at Fenwick for a little more than a year when she discovered this theorem on her calculator. Although she has been using English for only a couple of years, her proof of the theorem shows that mathematics is a universal language.

Catherine Mulligan
